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LETTER TO THE EDITOR

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Abstract. We define the intelligent spin states as those states which satisfy the Heisenberg equality for the spin operators: $\Delta J_x^2 \, \Delta J_y^2 = |\langle J_z \rangle|^2$. We find explicitly the $2j+1$ states which behave intelligently in each angular momentum space of spin j. For this purpose we use the Radcliffe states, showing that only the real and the pure imaginary Radcliffe states are intelligent. These intelligent states also satisfy the quartic consistency condition.

Our result, however, does not disagree in principle with the recent claim of Kolodziejczyk and Ryter that $|\mu\rangle = |0\rangle$ is the only state which minimizes the uncertainty product because minimum uncertainty does not necessarily imply intelligence.

Some time ago, Radcliffe (1971) defined for each $2j+1$ dimensional space H_j belonging to each finite irreducible representation of the rotation group the family of states $|\mu\rangle \equiv (1+|\mu|^2)^{-j} \exp(\mu J_-)|j\rangle$, where the parameter μ runs through the complex plane without restrictions. These states were obtained pushing out the analogy between the usual coherent states introduced by Senitzky (1958) (the eigenvectors $|z\rangle$ of the destruction boson operator a, ie $a|z\rangle = z|z\rangle$) and the exponential of the spin annihilation operator J_-. It turned out that this family $\{|\mu\rangle : \mu \in \mathbb{C}\}$ constitutes an overcomplete set in each H_j where they have been defined.

We want to give here some results concerning the solution of the non-linear problem of finding states $|w\rangle$ which verify the Heisenberg equality for the spin operators (J_x, J_y, J_z):

$$(\Delta J_x)^2_w (\Delta J_y)^2_w = \tfrac{1}{4}\langle w|J_z|w\rangle^2. \tag{1}$$

From here on we shall call the states $|w\rangle$ which satisfy equation (1) the 'intelligent' spin states. And the states $|m\rangle$ which minimize the quartic functional

$$\langle m|(\Delta J_x)^2|m\rangle \, \langle m|(\Delta J_y)^2|m\rangle \equiv F(m)$$

are going to be called minimum uncertainty states. In this letter we are going to give the explicit expression for the $(2j+1)$ states belonging to each H_j which verify equation (1).

It is well known (Louisell 1973) that all the intelligent spin states are contained in the set of states which solves the linear eigenvalue problem:

$$(J_x - \langle J_x \rangle \mathbb{1})|w\rangle = \mathrm{i}\alpha(J_y - \langle J_y \rangle \mathbb{1})|w\rangle, \tag{2}$$

with α a real number. This is equivalent to find the eigenvectors of the non-Hermitian operator $J_\alpha \equiv J_x - \mathrm{i}\alpha J_y$:

$$J_\alpha |w\rangle = w|w\rangle, \tag{3a}$$

along with the consistency condition

$$\langle w|J_\alpha w\rangle = w\langle w|w\rangle. \tag{3b}$$

Defining the real quantities $\gamma_\pm \equiv \frac{1}{2}(1\pm\alpha)$ the eigenvalue equation (3*a*) can be written in the form $J_\alpha|w\rangle \equiv (\gamma_+J_- + \gamma_-J_+)|w\rangle = w|w\rangle$ in which we introduced the standard ladder operators $J_\pm \equiv J_x \pm \mathrm{i}J_y$.

By employing the Radcliffe states we looked for a solution of the type ($E(j)$ is the integer part of the spin j):

$$|w\rangle = \sum_{l=0}^{l=N} a_l J_-^l|\mu\rangle, \qquad 0 \leqslant N \leqslant E(j). \tag{4}$$

After introducing this tentative solution in equation (3*a*) we arrived at the explicit solution of this proper value problem:

$$\mu_\pm = \pm(\gamma_+/\gamma_-)^{1/2} = \pm(1+\alpha/1-\alpha)^{1/2}, \tag{5a}$$

$$w_{N\pm} = 2(j-N)\mu_\pm\gamma_- = \pm(j-N)(1-\alpha^2)^{1/2}, \tag{5b}$$

$$|w_{N\pm}\rangle = a_0 \sum_{l=0}^{l=N} \binom{N}{l}\frac{(2j-l)!}{2j!}(-2\mu_\pm J_-)^l|\mu_\pm\rangle. \tag{5c}$$

In particular it is worth pointing out that for $N=0$ we obtained $|w_{0\pm}\rangle \equiv |\mu_\pm\rangle$. As $\mu_\pm^2$ is a real number ($\mu_\pm^2 = \gamma_\pm\gamma_\mp^{-1}$), $\mu_\pm$ is either real or pure imaginary. So, not every Radcliffe state is an intelligent state, only those Radcliffe states located on the real line or the imaginary axis† are intelligent states. Moreover there are intelligent states (for $N \neq 0$) which are not pure Radcliffe states.

Being aware that the expectation values between Radcliffe states of any operator defined on H_j might not coincide with the operator kernels, as Lieb (1973) pointed out; we verified the consistency equation (3*b*) for the $2j+1$ intelligent spin states.

Thereafter we can check whether the quartic homogeneous consistency condition (3*b*) is verified by the $2j+1$ $|w_{N\pm}\rangle$ states.

In order to make this calculation and normalize the states $|w_{N\pm}\rangle$ we used the fact that (λ, μ reals):

$$(J_-^{l_1}|\mu\rangle|J_-^{l_2}|\lambda\rangle) = (1+\lambda^2)^{-j}(1+\mu^2)^{-j}\partial_\mu^{l_1}\partial_\lambda^{l_2}[(1+\lambda\mu)^{2j}]. \tag{6}$$

It is interesting to show what are the relevant expectation values for $|w_{0\pm}\rangle = |\mu_\pm\rangle$. Using the results already given by Radcliffe for the values of $\langle J_x\rangle$, $\langle J_y\rangle$ and $\langle J_z\rangle$:

$$\langle J_x\rangle_\pm = \frac{2j\,\mathrm{Re}\,\mu}{1+|\mu|^2}, \qquad \langle J_y\rangle_\pm = \frac{2j\,\mathrm{Im}\,\mu}{1+|\mu|^2}, \tag{7a}$$

$$\langle J_z\rangle_\pm = j\left(\frac{1-|\mu|^2}{1+|\mu|^2}\right) = \{-\alpha j(|\alpha| \leqslant 1), -\alpha^{-1}j(|\alpha| \geqslant 1)\}, \tag{7b}$$

the quadratic quantities for the state $|w_{0\pm}\rangle$ are given by

$$(\Delta J_x)_\pm^2 \equiv \langle w_{0\pm}|(J_x - \langle J_x\rangle)^2|w_{0\pm}\rangle = \{\tfrac{1}{2}j\alpha^2(|\alpha| \leqslant 1); \tfrac{1}{2}j(|\alpha| \geqslant 1)\} \tag{7c}$$

$$(\Delta J_y)_\pm^2 = \{\tfrac{1}{2}j(|\alpha| \leqslant 1); \tfrac{1}{2}j\alpha^{-2}(|\alpha| \geqslant 1)\}, \tag{7d}$$

which obviously verify the Heisenberg equality.

† $\mu_\pm = \pm(1+\alpha)^{1/2}(1-\alpha)^{-1/2}$ ranges over the whole real axis or over the full imaginary axis according to whether $|\alpha|$ is less than or greater than 1 respectively.

A detailed account of these results, and the connection between these intelligent spin states and the 'coherent' non-compact states of SO(2, 1) found by Barut and Girardello (1971), and applications to study physical properties of some simple systems shall be given elsewhere.

Finally we want to mention that recently, Kolodziejczyk and Ryter (1974) claimed that $|\mu = 0\rangle$ is the only minimum uncertainty state for the SO(3) algebra. Their results do not contradict ours, because such kind of states $|m\rangle$, which minimize the homogeneous quartic functional $\langle m|\Delta J_x^2|m\rangle\langle m|\Delta J_y^2|m\rangle$ are not necessarily states which verify the Heisenberg equality.

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